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Affine Connection in Hilbert Space

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Abstract

A concise geometrodynamic approach to quantum theory is introduced, via a "quantum connection" Q_μ, which is the affine connection in Hilbert space. It is emphasized that this is the simplest and most natural interpretation of quantum mechanics in general relativity and yet has been largely neglected, so that much work remains to be done on it. The generalized Hilbert space has a simple Hermitian metric, but the precise form of Q_μ remains to be determined. The "quantum connection" is mathematically analogous to the spinor connection, which is discussed here for that reason, although the spinor connection arises in the first quantization, whereas Q_μ geometrizes the second quantization.

1. *Introduction*

It is well known that general relativity can be formulated on a nonholonomic basis (Misner et al., 1973; Weinberg, 1972; Petrov, 1969), on which the space-time metric retains the Lorentz form (the same as in special relativity), and the space-time connection (the affine connection of the space-time manifold) can be expressed in terms of an orthonormal tetrad or vierbein $e_\alpha{}^\mu$, where the superscript μ is a contravariant vector index, and the subscript α is the tetrad index. The $e_\alpha{}^\mu$ have covariant derivatives,

$$e^{\mu}_{\alpha;\nu} = e^{\mu}_{\alpha,\nu} + \Gamma^{\mu}_{\nu\sigma} e_\alpha{}^\sigma - \omega^{\beta}_{\alpha\nu} e_\beta{}^\mu \tag{1.1}$$

using summation convention over repeated indices (μ, ν, σ for tensors, and α, β, γ for tetrad indices), where $e^{\mu}_{\alpha,\nu} \equiv \partial e_\alpha{}^\mu / \partial x^\nu$ (partial derivatives with respect to space-time coordinates x^ν), the $\Gamma^{\mu}_{\nu\sigma}$ are Christoffel symbols (space-time connection coefficients on a holonomic basis), and the $\omega^{\beta}_{\alpha\nu}$ are the connection coefficients on the nonholonomic basis.

At first thought, equation (1.1) could seem wrong, because α is often called a "scalar" index, in so-called "covariant" derivatives which omit $\omega^{\beta}_{\alpha\nu}$. However, it must be emphasized that α is not a scalar index, and derivatives which neglect the $\omega^{\beta}_{\alpha\nu}$ are not covariant. Such misinterpretations obscure the useful role of α, β, γ as "Lorentz tensor" indices (Krause, 1975), subject to local Lorentz transformations, much as μ, ν, σ are subject to general coordinate transformations (Weinberg, 1972).

The Lorentz metric, $\eta_{\alpha\beta}$, has vanishing covariant derivatives,

$$\eta_{\alpha\beta;\mu} = -\omega^{\gamma}_{\alpha\mu}\eta_{\gamma\beta} - \omega^{\gamma}_{\beta\mu}\eta_{\gamma\alpha} \tag{1.2a}$$

$$= 0 \tag{1.2b}$$

imposing constraints on the $\omega^{\beta}_{\alpha\mu}$, and assuring that they have the properties of pseudo-Riemannian connection coefficients (Helgason, 1962).

The $\Gamma^{\mu}_{\nu\sigma}$ and $\omega^{\beta}_{\alpha\mu}$ are part of the classical theory of relativity, in which no thought is given to quantization.

When the first quantization is introduced, the spinor connection (Luehr, 1974; Weinberg, 1972) is essential for a covariant treatment of the Dirac equation.

When the second quantization (Roman, 1965) is introduced (hypothesizing complete quantization of general relativity), a "quantum connection", defined as the affine connection in the Hilbert space of the quantum mechanical state vector, is a natural vehicle for developing a theory of quantum geometrodynamics.

The spinor and space-time connections are mathematically analogous to the quantum connection, and the analysis is simplified by the use of a nonholonomic basis.

The analogous properties of the spinor connection are discussed, and then the quantum connection is introduced.

2. *Spinor Connection*

Equation (1.2) expresses the basic principle that the metric is an absolute constant. Similarly, the Dirac matrices, γ^{α}, are absolute constants (Audretsch, 1974), and, when expressed on a nonholonomic basis, have the same form as in special relativity.

It is convenient to express the spinor connection, Ω_{μ}, in a closed form, as a 4 x 4 matrix, in terms of which the γ^{α} have vanishing covariant derivatives,

$$\gamma^{\alpha}_{;\mu} = \gamma^{\beta}\omega^{\alpha}_{\beta\mu} + \Omega_{\mu}\gamma^{\alpha} - \gamma^{\alpha}\Omega_{\mu} \tag{2.1a}$$

$$= 0 \tag{2.1b}$$

relating Ω_{μ} to the $\omega^{\alpha}_{\beta\mu}$, and indicating that γ^{α} is a mixed second-rank spinor, as well as a contravariant "Lorentz vector" on the tetrad index α (= 0, 1, 2, 3).

The matrix Ω_{μ} has complex coefficients, and a Hermitian conjugate $\Omega^{\dagger}_{\mu}$. As in special relativity, the γ^{α} can be chosen so that $(\gamma^0)^{\dagger} = \gamma^0$, and $(\gamma^i)^{\dagger} = -\gamma^i$, for $i = 1, 2, 3$.

The spinor metric β then has the simple Hermitian form,

$$\beta = \gamma^0 \tag{2.2}$$

and a vanishing covariant derivative,

$$\beta_{;\mu} = -\beta\Omega_\mu - \Omega_\mu^\dagger\beta \tag{2.3a}$$

$$= 0 \tag{2.3b}$$

satisfying the requirement $(\beta_{;\mu})^\dagger = \beta_{;\mu}$, and indicating that β is a covariant second-rank spinor (just as $\eta_{\alpha\beta}$ is a covariant second-rank "Lorentz tensor").

At first thought, equation (2.2) could seem wrong, because it equates quantities that have different transformation properties. What this means, of course, is that equation (2.2) is not covariant. It holds only on the non-holonomic basis used here. On a holonomic basis, the spinor and space-time metrics can be constructed from the Dirac matrics, but the relationships are more complicated, thus indicating one of the advantages of non-holonomic representations.

For any matrix $\mathcal{O}$ acting as a linear operator in spinor space, it is useful to define the adjoint,

$$\not{\mathcal{O}} \equiv \beta^{-1}\mathcal{O}^\dagger\beta \tag{2.4}$$

Equation (2.3) $\Rightarrow \not{\Omega}_\mu = -\Omega_\mu$, i.e., Ω_μ is skew-adjoint. It follows from the properties of the Dirac matrices that $\not{\gamma}^\alpha = \gamma^\alpha$, i.e., the γ^α are self-adjoint, which is the property of "observables" in this formalism (observables are not represented by Hermitian operators unless the metric is Euclidean).

γ^α represents the 4-velocity of the spinor particle. Introducing a 4-component spinor field φ, using geometrized units (Misner et al., 1973) with $i \equiv (-1)^{1/2}$, and defining a 4-momentum operator,

$$p_\alpha \equiv ie_\alpha{}^\mu(\partial_\mu + \Omega_\mu) \tag{2.5}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$, the generalized Dirac equation for a free particle of proper mass m, expressed on a nonholonomic basis, is

$$\gamma^\alpha p_\alpha\varphi = m\varphi \tag{2.6}$$

representing a quantum mechanical statement of the relationship between 4-velocity and 4-momentum, expressed in the language of the first quantization, in which Ω_μ is the affine connection for the Hilbert space of φ.

A similar treatment is possible for the second quantization.

3. *Quantum Connection*

In the "Heisenberg picture" of non-relativistic and special-relativistic quantum theory (Roman, 1965), the second quantized state vector ψ is constant, i.e., independent of the space-time coordinates x^μ ($\mu = 0, 1, 2, 3$), so that $\psi_{,\mu} = 0$.

In general relativity, in accordance with the "simplicity principle", which seeks the simplest and most direct generalization from the special to the general theory (Adler et al., 1965), this result should have the form

$$\psi_{;\mu} = 0 \tag{3.1}$$

expressed in terms of a covariant derivative

$$\psi_{;\mu} = \psi_{,\mu} + Q_\mu \psi \tag{3.2}$$

where Q_μ is the "quantum connection", i.e., a linear operator whose components function as affine connection coefficients (Eisenhart, 1928) in the second-quantized Hilbert space (the infinite-dimensional complex vector space of ψ).

The general relativistic Hilbert space (a generalization of the usual Hilbert space) is assumed to have a Hermitian metric $h = h^\dagger$. In this respect, the Hilbert space is pseudounitary (Schouten, 1954), sometimes called unitary (just as pseudo-Riemannian spaces are sometimes called Riemannian).

If ψ is constructed as a column matrix, so that the Hermitian conjugate $\psi^\dagger$ is a row matrix, then ψ adjoint can be defined as the row matrix

$$\not{\psi} \equiv \psi^\dagger h = (h\psi)^\dagger \tag{3.3}$$

Equations (3.2)–(3.3) then give the covariant derivative,

$$\not{\psi}_{;\mu} = \not{\psi}_{,\mu} + \not{\psi}\not{Q}_\mu \tag{3.4}$$

with

$$\not{Q}_\mu \equiv h^{-1} Q_\mu^\dagger h \tag{3.5}$$

the adjoint of Q_μ, where h^{-1} is the inverse of h.

For a normalized vector, $\not{\psi}\,\psi = 1$, and the pure-state density operator is

$$\rho = \psi\,\not{\psi} \tag{3.6}$$

Equations (3.5) and (3.6) $\Rightarrow \rho = \not{\rho} \equiv h^{-1}\rho^\dagger h$, i.e., ρ is self-adjoint.

If ψ is defined as a "contravariant" h vector (abbreviation for Hilbert space vector), then $\not{\psi}$ is a "covariant" h vector, and ρ is a mixed second-rank h tensor (Hilbert space tensor).

Another mixed second-rank h tensor is the identity operator,

$$I = h^{-1}h = hh^{-1} \tag{3.7}$$

the mixed form of the metric.

Since the metric is an "absolute constant", it follows that I has vanishing covariant derivatives,

$$I_{;\mu} = Q_\mu + \not{Q}_\mu = 0 \tag{3.8}$$

indicating that Q_μ is skew-adjoint, in analogy to Ω_μ.

Equations (3.6) and (3.8) give the covariant derivative of ρ,

$$\rho_{;\mu} = \rho_{,\mu} + Q_\mu \rho - \rho Q_\mu \tag{3.9}$$

noting the evident similarity between the commutator formalism (of quantum mechanics) and the connection formalism (of affine spaces). Equation (3.1) $\Rightarrow \rho_{;\mu} = 0$, so equation (3.9) is a dynamical equation for ρ.

A pseudounitary transformation U is defined as a linear operator preserving the metric h through the isometry,

$$U^\dagger h U = h \tag{3.10}$$

equivalent to

$$\not{U} = U^{-1} \tag{3.11}$$

indicating how U is a straightforward generalization of a unitary operator.

It is a basic hypothesis that the symmetry transformations, i.e., the dynamical symmetries under which the generally covariant field equations are form invariant, are isometries of h, so that the linear symmetry transformations have a pseudounitary representation in Hilbert space, while the remaining symmetries (those involving time inversion) (Wigner, 1959) are antilinear isometries.

Under a space-time coordinate transformation expressible as a diffeomorphism,

$$\partial_\mu \to \lambda_\mu{}^\nu \partial_\nu \tag{3.12}$$

where λ is the space-time representation of the symmetry, the corresponding Hilbert space representation is the automorphism,

$$\psi \to U\psi \tag{3.13}$$

where U is pseudounitary.

There are then the associated transformations

$$\rho \to U\rho U^{-1} \tag{3.14a}$$

$$\psi_{;\mu} \to \lambda_\mu{}^\nu U\psi_{;\nu} \tag{3.14b}$$

$$\rho_{;\mu} \to \lambda_\mu{}^\nu U\rho_{;\nu} U^{-1} \tag{3.14c}$$

$$Q_\mu \to \lambda_\mu{}^\nu U Q_\nu' U^{-1} \tag{3.14d}$$

where

$$Q_\nu' \equiv Q_\nu - U^{-1} U_{,\nu} \tag{3.15}$$

Equations (3.14a)-(3.14c) are tensor transformations, while equations (3.14d) and (3.15) indicate how Q_μ differs from a tensor when $U = U(x)$, i.e., when U has a functional dependence on the x^μ.

The Hilbert space metric, h, is expressed in a doubly covariant form, which has the same transformation property as the dyad, $\not{\psi}^\dagger \not{\psi}$. Equations (3.4) and (3.8) then give the covariant derivative,

$$h_{;\mu} = h_{,\mu} - Q_\mu^\dagger h - hQ_\mu \tag{3.16}$$

while equations (3.5) and (3.8) give

$$Q_\mu^\dagger h = -hQ_\mu \tag{3.17}$$

Since h is an absolute constant, equations (3.16) and (3.17) give

$$h_{;\mu} = h_{,\mu} = 0 \tag{3.18}$$

indicating that h cannot have any dependence on the x^μ.

Consequently, without loss of generality, h can be chosen to have a diagonal form, with eigenvalues ± 1, so that

$$h = h^\dagger = \bar{h} = h^{-1} \tag{3.19}$$

where h is now Hermitian ($h^\dagger = h$), unitary ($h^\dagger = h^{-1}$), self-adjoint ($\bar{h} = h$), and pseudounitary ($\bar{h} = h^{-1}$).

In other words, h can be chosen to have the same form as in special relativity, where Hilbert space formalisms have already been constructed with an indefinite metric (Nagy, 1966).

The difference, then, is in $Q_\mu \equiv iP_\mu$, where $P_\mu = \bar{P}_\mu$, so that P_μ is self-adjoint, has the units of 4-momentum, and can in fact be identified with a total 4-momentum operator in special relativity (Epstein, 1971), though not in general relativity.

In special relativity, just as the space-time connection coefficients (the $\Gamma^\mu_{\nu\sigma}$ and the $\omega^\beta_{\alpha\mu}$) and the spinor connection (Ω_μ) vanish in an inertial frame, so Q_μ vanishes in the Heisenberg picture (the quantum analog of an inertial frame). In the Schrödinger picture, on the other hand, P_0 becomes a Hamiltonian operator, reintroduced by equations (3.14d) and (3.15).

In general relativity, just as inertial frames are at best local, the Heisenberg picture is at best a local picture, in that Q_μ may vanish locally (at some values of the x^μ), but not globally (i.e., not throughout all space-time). Q_μ must then be determined from basic considerations, mathematically analogous to equation (1.1) (which relates the $\omega^\beta_{\alpha\mu}$ to the $\Gamma^\mu_{\nu\sigma}$), or equation (2.1) (which relates Ω_μ to the $\omega^\beta_{\alpha\mu}$), or equally fundamental relationships entailed by the topological group structure of the affine spaces which have been hypothesized to formulate a relativistically covariant theory of quantum geometrodynamics.

4. *Conclusions*

The determination of the "quantum connection" Q_μ is an important unsolved problem in relativistic quantum field theory, and involves a natural generalization of the Hamiltonian formalism (Ben-Abraham and Lonke, 1973). Despite the similarity of Q_μ to Ω_μ, the quantum connection has been approached only in rudimentary ways, while the spinor connection has been exhaustively analyzed.

The approach here is not so much a quantization of general relativity as it is a geometrization of quantum theory, which Q_μ expresses in the language of geometrodynamics.

Absolute constants, having different types of relativistic indices, are useful for relating the different types of affine connections. Such quantities having h-tensor indices are to be sought, as a means of constructing Q_μ, like equation (2.1) constructs Ω_μ, in terms of known functions.

References

Adler, R., Bazin, M., and Schiffer, M. (1965). *Introduction to General Relativity.* (McGraw-Hill, New York).

Audretsch, J. (1974). *International Journal of Theoretical Physics*, 9, 323.
Ben-Abraham, S. I., and Lonke, A. (1973). *Journal of Mathematical Physics*, **14**, 1935.
Eisenhart, L. P. (1928). *Non-Riemannian Geometry.* (American Mathematical Society, Providence, Rhode Island).
Epstein, K. J. (1971). *Journal of Mathematical Physics*, **12**, 125.
Helgason, S. (1962). *Differential Geometry and Symmetric Spaces.* (Academic Press, New York), pp. 47–55.
Krause, J. (1975). *International Journal of Theoretical Physics*, **12**, 35.
Luehr, C. P. (1974). *Journal of Mathematical Physics*, **15**, 1120.
Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation.* (Freeman, San Francisco), pp. 36, 210.
Nagy, K. L. (1966). *State Vector Spaces with Indefinite Metric in Quantum Field Theory.* (Noordhoff, Groningen, Netherlands).
Petrov, A. Z. (1969). *Einstein Spaces.* (Pergamon Press, Oxford), pp. 279–86.
Roman, P. (1965). *Advanced Quantum Theory.* (Addison-Wesley, Reading, Massachusetts), pp. 45, 63, 359.
Schouten, J. A. (1954). *Ricci Calculus*, 2nd Edn. (Springer, Berlin).
Weinberg, S. (1972). *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* (Wiley, New York), pp. 365–73.
Wigner, E. P. (1959). *Group Theory.* (Academic Press, New York).